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FINITE VOLUME CHARACTERISTIC FLUX SCHEME FOR TRANSONIC FLOW PROBLEMS

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Abstract

This work deals with the numerical solution of internal transonic flow problems. Currently we expect 1D and 2D inviscid flow of perfect gas modelled by the Euler equations. We use finite volume characteristic flux scheme, called VFFC scheme, which can be viewed as a generalization of Roe scheme. The dissipation matrix is computed analytically or numerically. We present numerical results for 1D shock tube problem computed by the second order method, where the spatial accuracy is due to linear reconstruction with the minmod limiter and temporal discretization is done using explicit three stage Runge-Kutta method. Further numerical results for 2D transonic flow in GAMM channel have been achieved by the first order method on structured quadrilateral as well as unstructured triangular meshes.

Introduction

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VFFC scheme for 2D system of nonlinear transport equations

Let us consider two-dimensional system of nonlinear transport equations in form

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{W})}{\partial y} = \mathbf{0} \tag{1}$$

where $\mathbf{W}(x,t):\ \mathbb{R}^2 \to \mathbb{R}^n$ and $\mathbf{F}(\mathbf{W}), \mathbf{G}(\mathbf{W}):\ \mathbb{R}^n \to \mathbb{R}^n$ are vectors. Now integrate this Eq. (1) along the interval $\langle t_0; t_0+\Delta t\rangle$ and the subset $K \subset \mathbb{R}^2$ (eg. triangle or quadrilateral)

$$\int\limits_{t_0}^{t_0+\Delta t} \int\limits_K \left(\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{W})}{\partial y} \right) dK dt = \mathbf{0}. \tag{2}$$

Applying the mean value and the Green's theorems we get a common expression

$$vol(K)\left[\mathbf{W}_K(t_0+\Delta t) - \mathbf{W}_K(t_0)\right] = -\Delta t \oint\limits_{\partial K} \left[\mathbf{F}(t_0+\Delta t/2)n_x + \mathbf{G}(t_0+\Delta t/2)n_y\right] ds \tag{3}$$

where (n_x, n_y) denotes the outer unit normal of ∂K. The line integral can be approximated by the sum of N_{edg} contributions for edges of K

$$\left[\mathbf{W}_K(t_0+\Delta t) - \mathbf{W}_K(t_0)\right] = -\frac{\Delta t}{vol(K)} \sum_{j=1}^{N_edg} \left[\mathbf{F}_j(t_0+\Delta t/2)n_{x,j} + \mathbf{G}_j(t_0+\Delta t/2)n_{y,j}\right] \Delta s_j, \tag{4}$$

where Δs_j is the length of j-th edge. The main point is the evaluation of normal flux $\mathbf{F}_j(t_0+\Delta t/2)n_{x,j} + \mathbf{G}_j(t_0+\Delta t/2)n_{y,j}$. Let's go back to the Eq. (1) and consider a new coordinate n_j, which is perpendicular to the j-th cell edge, and which is equal to zero just on the edge and positive in the direction of outer normal $n_j = n_{j,0} + (n_{x,j}, n_{y,j})(x,y) = n_{0,j} + n_{x,j}x + n_{y,j}y$, where $n_{0,j}$, $n_{x,j}$ and $n_{y,j}$ are constants. Further we will omit the edge index j. The Eq. (1) can be written as

$$\frac{\partial \mathbf{W}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{W})}{\partial n}\frac{\partial n}{\partial x}+\frac{\partial \mathbf{G}(\mathbf{W})}{\partial n}\frac{\partial n}{\partial y}=\frac{\partial \mathbf{W}}{\partial t}+\frac{\partial\Big(\mathbf{F}(\mathbf{W})n_x+\mathbf{G}(\mathbf{W})n_y\Big)}{\partial n}=\frac{\partial \mathbf{W}}{\partial t}+\frac{\partial \mathbf{F}_n(\mathbf{W})}{\partial n}=\mathbf{0} \quad (5)$$

and introducing the Jacobian $\mathbf{J}_n(\mathbf{W})=\partial\mathbf{F}_n/\partial\mathbf{W}$ we get

$$\mathbf{J}_n(\mathbf{W})\frac{\partial \mathbf{W}}{\partial t}+\mathbf{J}_n(\mathbf{W})\frac{\partial \mathbf{F}_n(\mathbf{W})}{\partial x}=\frac{\partial \mathbf{F}_n(\mathbf{W})}{\partial t}+\mathbf{J}_n(\mathbf{W})\frac{\partial \mathbf{F}_n(\mathbf{W})}{\partial x}=\mathbf{0}, \quad (6)$$

i.e. that transport equation for flux $\mathbf{F}_n$ is identical with the one for $\mathbf{W}$. The VFFC flux [1] is

$$\mathbf{\Phi}_n(\mathbf{W}^-,\mathbf{W}^+,\mu)=\frac{\mathbf{F}_n(\mathbf{W}^-)+\mathbf{F}_n(\mathbf{W}^+)}{2}-\mathbf{U_n}(\mu)\frac{\mathbf{F}_n(\mathbf{W}^+)-\mathbf{F}_n(\mathbf{W}^-)}{2}, \quad (7)$$

where $\mathbf{W}^-$ and $\mathbf{W}^+$ are the states for negative and positive n respectively, the dissipation matrix $\mathbf{U}_n=sign(\mathbf{J}_n)=sign(\mathbf{R\Lambda L})=\mathbf{R}sign(\mathbf{\Lambda})\mathbf{L}$, where sign matrix $sign(\mathbf{\Lambda})$ we get from diagonal matrix $\mathbf{\Lambda}$ replacing negative and positive numbers by -1 and 1 respectively. The sign matrix $sign(\mathbf{J}_n)$ can be computed also numerically directly from matrix $\mathbf{J}_n$, i.e. without eigenvectors decomposition, see [3]. Variable μ is the average $\mu=(vol^-\mathbf{W}^-+vol^+\mathbf{W}^+)/(vol^-+vol^+)$, where vol^- and vol^+ are volumes of cells sharing the interface for n negative and positive respectively. The first order VFFC scheme in 2D can be written as

$$\mathbf{W}_K(t_0+\Delta t)=\mathbf{W}_K(t_0)-\frac{\Delta t}{vol(K)}\sum_{j=1}^{4}\mathbf{\Phi}_{n_j}\Big(\mathbf{W}_K(t_0),\mathbf{W}_j^+(t_0),\mu_j\Big)\Delta s_j, \quad (8)$$

where $\mathbf{W}_j^+$ is the $\mathbf{W}$ in the cell, which is sharing the j-th edge with cell K.

2D Euler equations

Similarly, the 2D flow of compressible inviscid fluid can be modelled by the 2D Euler equations

$$\mathbf{W}=\begin{bmatrix}\rho\\ \rho u\\ \rho v\\ E\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(E+p)\end{bmatrix},\quad \mathbf{G}=\begin{bmatrix}\rho v\\ \rho vu\\ \rho v^2+p\\ v(E+p)\end{bmatrix} \quad (9)$$

where u and v are velocity components and the rest of variables has the same meaning like in 1D case. The system is closed by the equation of pressure (perfect gas) $p=(\gamma-1)(E-\rho(u^2+v^2)/2)$. To simplify the computation of normal flux in the Eq. (8) we consider the rotation which aligns the unit vector $\vec{n}_j(n_{x,j},n_{y,j})$ with axis x, so one get after rotation $\vec{\tilde{n}}_j=(\tilde{n}_{x,j},\tilde{n}_{y,j})=(1,0)$. The rotation can be expressed in form of matrix-vector product

$$\begin{bmatrix}\tilde{n}_{x,j}\\ \tilde{n}_{y,j}\end{bmatrix}=\begin{bmatrix}r_{1,1} & r_{1,2}\\ r_{2,1} & r_{2,2}\end{bmatrix}\begin{bmatrix}n_{x,j}\\ n_{y,j}\end{bmatrix} \quad (10)$$

Scalar quantities are invariant to the rotation. Now denote by $\tilde{\mathbf{W}}$ a vector of unknown after rotation. It is possible to write $\tilde{\mathbf{W}}=\mathbf{TW}$, $\quad\mathbf{W}=\mathbf{T}^{-1}\tilde{\mathbf{W}}$ where

$$\mathbf{T}=\begin{bmatrix}1&0&0&0\\0&r_{1,1}&r_{1,2}&0\\0&r_{2,1}&r_{2,2}&0\\0&0&0&1\end{bmatrix},\quad \mathbf{T}^{-1}=\begin{bmatrix}1&0&0&0\\0&r_{1,1}&-r_{1,2}&0\\0&-r_{2,1}&r_{2,2}&0\\0&0&0&1\end{bmatrix} \quad (11)$$

The Eq. (8) can be then written as

$$\mathbf{W}_K(t_0+\Delta t)=\mathbf{W}_K(t_0)-\frac{\Delta t}{vol(K)}\sum_{j=1}^{4}\mathbf{T}_j^{-1}\mathbf{\Phi}_{n_j}\Big(\tilde{\mathbf{W}}_K(t_0),\tilde{\mathbf{W}}_j^+(t_0),\mu_j\Big)\Delta s_j, \quad (12)$$

where $n_{x,j} = 1$ and $n_{y,j} = 0$, so $\mathbf{F}_n = \mathbf{F}$ and

$$\mathbf{J}_n(\mathbf{W}) = \frac{\partial \mathbf{F}}{\partial \mathbf{W}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ (\gamma-3)\frac{u^2}{2} + (\gamma-1)\frac{v^2}{2} & (3-\gamma)u & (1-\gamma)v & \gamma-1 \\ -uv & v & u & 0 \\ u\left((\gamma-1)\frac{u^2+v^2}{2} - H\right) & H-(\gamma-1)u^2 & (1-\gamma)uv & \gamma u \end{bmatrix} \tag{13}$$

with

$$H = \frac{E+p}{\rho} = \frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1}, \qquad c^2 = \frac{\gamma p}{\rho}. \tag{14}$$

Matrix $\mathbf{J}_n(\mathbf{W})$ has following eigenvalues $\lambda_1 = u - c$, $\lambda_2 = u + c$, $\lambda_3 = u$ and $\lambda_4 = u$. The corresponding eigenvectors matrixes are

$$\mathbf{R} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ u-c & u+c & u & 0 \\ v & v & v & 1 \\ H-uc & H+uc & \frac{u^2+v^2}{2} & v \end{bmatrix}, \qquad \mathbf{L} = \mathbf{R}^{-1}. \tag{15}$$

1D shock tube test case (Sod test)

The test description can be found e.g. on [4]. The solution domain is rectangle with $x \in \langle 0; 1\rangle$ and $t \in \langle 0; T\rangle$. We solve the initial value problem with the initial data at $t = 0$

$$\rho(x,0) = \begin{cases} 1.000 & , 0 \le x < 0.5 \\ 0.125 & , 0.5 < x \le 1 \end{cases} \qquad p(x,0) = \begin{cases} 10 & , 0 \le x < 0.5 \\ 1 & , 0.5 < x \le 1 \end{cases} \tag{16}$$

and $u(x,0) = 0$. Specific heat ratio $\gamma = 1.4$ is expected. Following figures show the comparison of numerical results with analytical solution at time $t = 0.2$. The computational grid is uniform either with 100 or 1000 cells. The $2^n d$ order VFFC method includes the linear reconstruction with minmod limiter and 3-stage Runge-Kutta method for temporal discretization.

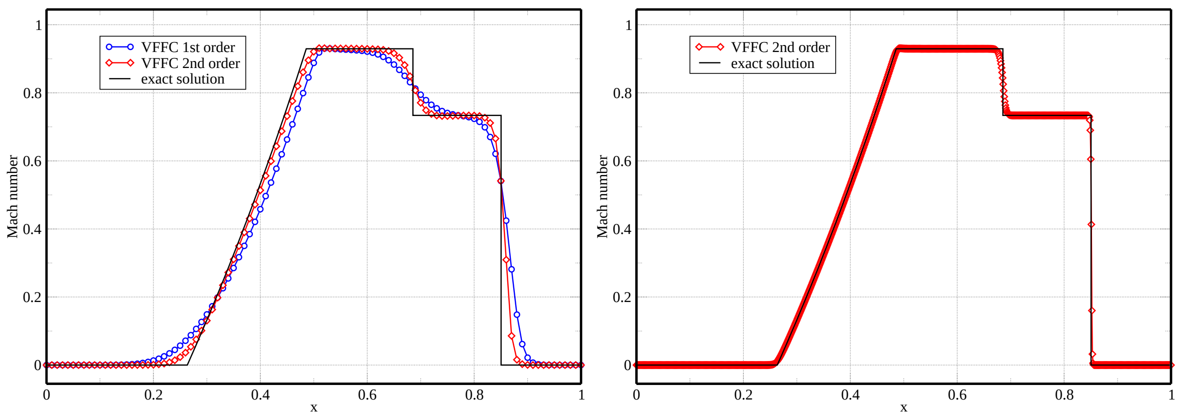

Figure 1: Mach number at $t = 0.2$, numerical results for 100 cells (left) and 1000 cells (right).

2D GAMM channel

The solution domain is rectangle $\langle -1; 2\rangle \times \langle 0; 1\rangle$ with circular bump (center $[0.5; -1.2]$, radius 1.3). The initial conditions are: $\rho = \rho_0 = 1$, $u = v = 0$ and $p = p_0 = 1/\gamma$, where $\gamma = 1.4$. The inlet boundary ($x = -1$) conditions are: $\rho_0 = 1$, $p_0 = 1/\gamma$ and $v = 0$. The outlet boundary ($x = 2$) condition is: $p/p_0 = 0.736952$. The rest of boundary is non-permeable wall, therefore we prescribe normal velocity component equal to zero. Figures show the numerical results of $1^s t$ order VFFC method for structured as well as unstructured grid.

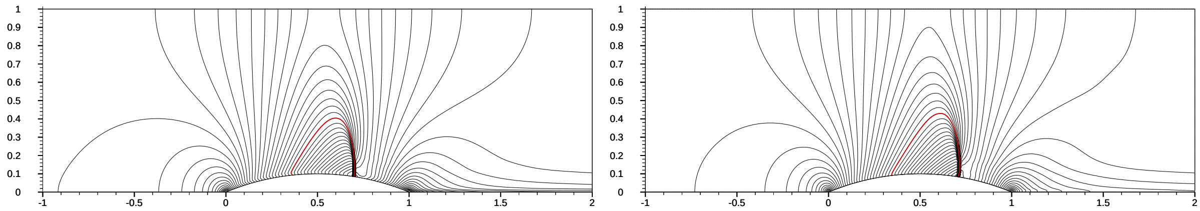

Figure 2: Mach number isolines ($\Delta M = 0.02$, bold line denotes $\Delta M = 1$), structured quadrilateral grid 450×150 cells (left) and unstructured grid with $1.1 \cdot 10^5$ triangles (right)

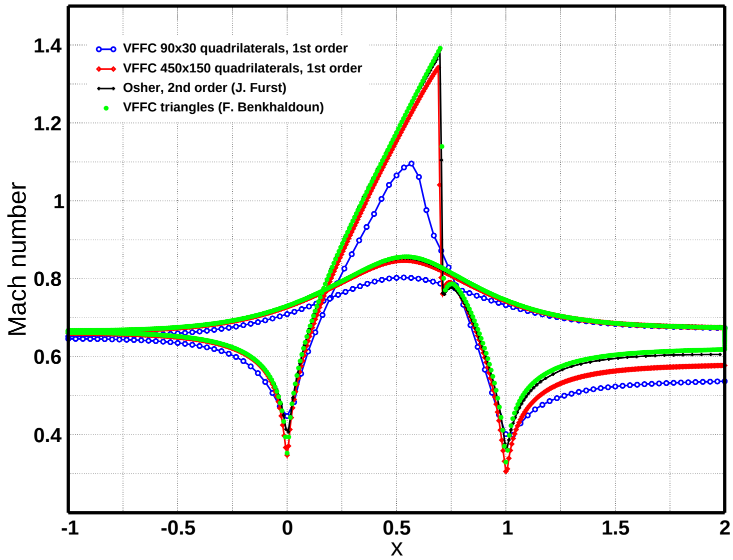

Figure 3: Mach number along lower and upper wall

Conclusions

The first results of internal transonic flow problems achieved by VFFC scheme are presented. The agreement between numerical results and analytical solution (unsteady flow - Sod test) as well as comparison with numerical results of other author is good and promising. The sign matrix in the case of Euler equations (real eigenvalues and complete set of eigenvectors) can be easily obtained in analytic form. The numerical computation of sign matrix, although it works well, is not so interesting in this simple case, since it brings some extra cost in term of CPU time. Future work is aimed at extension of presented method for the solution of two-phase flow problems with two immiscible compressible fluids.

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